SYNTHESIS OF RELIABLE TELECOMMUNICATION NETWORKS

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Summary In many application, the network designer may want to know to synthesise a reliable telecommunication network. Assume that a network, denoted as $G_{m,e}$, has the number of nodes n and the number of edges e, and the operational probability of each edge is known. The system reliability of the network is defined to be the reliability that every pair of nodes can communicate with each other. A network synthesis problem considered in this paper is to find a network $G^*_{n,e}$, that maximises system reliability over the class of all networks for the classes of networks $G_{n,n-1}$, $G_{n,n}$ and $G_{n,n+1}$ respectively. In addition an upper bound of maximum reliability for the networks with n-node and e-edge ($e \geq n+2$) is derived in terms of node degrees. Computational experiments for the reliability upper are also presented. The results show, that the proposed reliability upper bound is effective.

1. INTRODUCTION

In the design of telecommunication and computer network, system reliability is an important parameter. The network may want to know how to synthesise a network such that the network system reliability is maximised. In telecommunication and computer network, system reliability can be defined as the probability that every pair of nodes can communicate with each other. In contrast to the large literature on computing and analysis of network reliability, there are very few results concerned with synthesis problem.

In this paper we consider a network synthesis problem. Assume that the number of nodes n, the number of edges e, and the operational probability of each edge are know. The synthesis problem is to find a network that maximises system reliability over the class of all networks having n nodes and e edges. A network with n nodes and e edges is denoted as $G_{n,e}$. In this paper, we find the optimal networks for the classes of networks $G_{n,n-1}$, $G_{n,n}$ and $G_{n,n+1}$, respectively. In addition, an upper bound of maximum reliability for the networks with n-node and e-edge ($e \geq n+2$) is derived in terms of node degrees. Determining maximum reliability or an upper bound of maximum reliability of networks $G_{n,e}$ is useful for network designer because, in some applications, it is desired to find the least cost network under a given reliability constraint. Maximum reliability or its upper bound can be used to determine a lower bound of the minimum number of edges, in the network which may satisfy the reliability constraint.

A computer communication network can be modelled by probabilistic graph $G=(V, E, p)$, in which V and E are the set of nodes and edges that represent the computers and communication links, respectively. Thus, two terms, network and graph are used interchangeably in this paper. We assume that all nodes are perfectly reliable but any edge e, $e \in E$, may work (fail) with probability p ($q=1-p$). In edition, all edge failures are assumed to occur independently of each other. The system reliability of G, denote by $R(G)$, is the probability that all nodes in G are connected. Note that spanning trees of G, denoted as T_i is connect all nodes *in* G. The system reliability can then be written as

$$R(G) = P_r\{U_{i=1}^{n_{st}} A_i\}$$

where A_i is the event in which spanning tree T_i is operational and n_{st} is the number of spanning tree T_i is operational and n_{st} is the number of spanning trees in G. Thus, the problem considered in this paper con be mathematically stated as

$$R(G^*_{n,e}) = \max\{R(G_{n,e}) / G = (V, E, p), / V / =$$
$$= n, / E / = e\},$$

where $G^*_{n,e}$ is the graph that maximises system reliability over class of all graphs having n nodes and e edges. We call such a graph $G^*_{n,e}$ as the maximum reliability graph.

In section 2, we find the maximum reliability graph of $G_{n,n-1}$, $G_{n,n}$ and $G_{n,n+1}$. Section 3 derives the upper bounds of $R(G^*_{n,e})$ for $e=n+2,...,n(n-1)/2$. Computational results are presented in section 4 to show the effectiveness of the reliability upper bounds.

2. MAXIMAL SYSTEM RELIABILITY OF NETWORKS $G_{n,n-1}$, $G_{n,n}$ AND $G_{n,n+1}$

Lemma 1 Let n be the number of nodes in the network and p be the operation probability of each edge. Then,

$$1. R(G^*_{n,n}) = p^{n-1}$$

$$2. R(G^*_{n,n}) = p^n + (1-p).$$

Lemma 2 Let n be the number of nodes in the network an p be the operation probability of each edge. Let C^+_n be a cycle including a chord. Then

$$\max\{R(C_n^+)\} = \begin{cases} = p^n + np^{n-1}(1-p) + (\frac{n}{2})^2 p^{n-1}(1-p)^2 \\ if.n.is.even \\ = p^n + np^{n-1}(1-p) + (\frac{n^2-1}{4}) p^{n-1}(1-p)^2 \\ otherwise \end{cases}$$

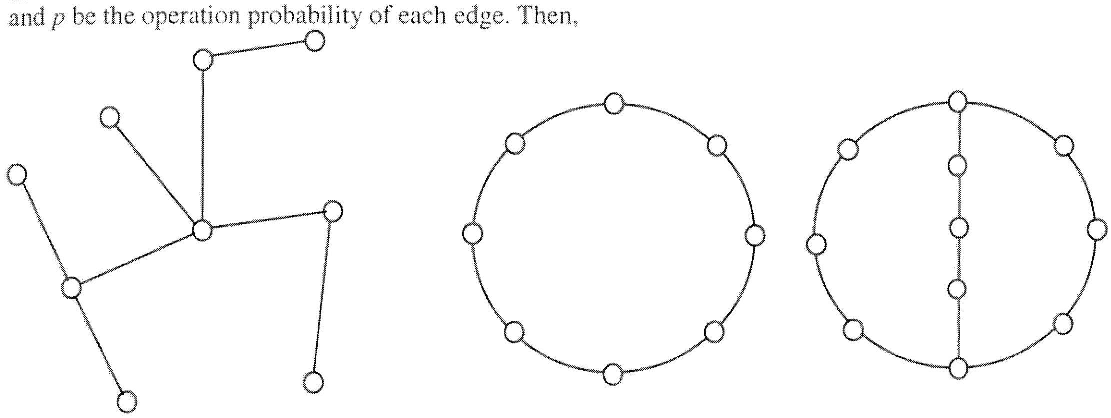
Lemma 3 Let $\widehat{G}_{n,n+1}$ be a graph with n nodes and $n+1$ edges and contain two elementary cycles without any common edge. Then

$$R(\widehat{G}_{n,n+1}) \langle \max\{R(C_n^+)\}$$

Lemma 4 Let n be the number of nodes in the network and p be the operation probability of each edge. Then,

$$R(G^*_{n,n+1}) = p^{n+1} + (n+1)p^n(1-p) + \frac{(n+1)^2}{3} p^{n-1}(1-p)^2$$

An interesting result from Lemma + is that the topology of $G^*_{n,n-1}$ and $G^*_{n,n}$ are the spanning tree and the ring respectively. Furthermore, Lemma 4 can be generalised. Then, we can obtain that the topology of $G^*_{n,n+1}$ three cycles and the lengths of these cycles do not differ by more than one. The topologies of $G^*_{n,n-1}, G^*_{n,n}$ and $G^*_{n,n+1}$ are shown in Fig. 1.



Spanning tree Cycle The graph with 3 cycles

Fig. 1 Examles of networks

3. UPPER BOUNDS OF MAXIMUM RELIABILITY OF NETWORKS $G_{n,e}$

Before stating the expressions for the upper bounds of $R(G_{n,l}$, $l=n+2, n+3,..., n(n-1)/2$, we show a method to compute an upper bound of reliability for the network having nodes of degree $d_1, d_2, ...,$ and d_n. The degree of node i is defined as the number of links incident with node i. Note that each network can be associated with a unique sequence of degrees called its degree sequence and

$$\sum_{i=1}^{n} d_i = 2l,$$

where l is total number of links in the network. For convenience, $d_1 \le d_2 \le ... \le d_n$ is assumed in this section. The node having degree d_i is labeled as node i. The minimal cutset of a connected graph G is a set of links whose from G leaves G disconnected, provided deletion of any proper subset of these links does not disconnect G. Let $\mu(i)$ be the set that contains all links incident to node i. then, $\mu(i)$ is cutset of G. Let C_j denote the event that all links in the j th minimal cutset fail and $\overline{C}_j$ the complement of this event. In addition, let F_i denote the event that all links incident to node i fail and F_i denote the event that links incident to node i hail and $\overline{F}_i$ denote complement of this event. By sum of product (SOP), therefore, the unreliability of the network G with degree sequence $d_1, d_2, ...,$ and d_n is

$$1 - R(G) = P_r(C_1) + P_r(C_2 \cap \overline{C}_1) + ... +$$
$$+ P_r(C_M \cap \overline{C}_1 \cap \overline{C}_2 \cap ... \cap C_{M-1}) \ge$$
$$\ge P_r(F_1) + P_r(F_2 \cap \overline{F}_1) + ... +$$
$$+ P_r(F_n \cap \overline{F}_1 \cap \overline{F}_2 \cap ... \cap \overline{F}_{n-1}).$$

Let $q=1-p$ and we have $P_r(F_1) = q^{d_1}$ an with recursive process we obtain

$$1 - R(G) \ge$$
$$\ge q^{d_1} + \sum_{i=2}^{n} q^{d_i} \prod_{k=1}^{m_i} (1 - q^{d_k - 1}) \prod_{k=m_i+1}^{i-1} (1 - q^{d_k}) =$$
$$= \sum_{i=1}^{n} q^{d_i} \prod_{k=1}^{m_i} (1 - q^{d_k - 1}) \prod_{k=m_i+1}^{i-1} (1 - q^{d_k})$$

where $m_i=min(d_i, i\text{-}1)$, $i=1, 2, ..., n$.

Lemma 5 let G be a network with degree sequences d_1, d_2, ... and d_n, and $q=1\text{-}p$ be the reliability of each link in G. Then the reliability of network G is

$$R(G)\leq$$

$$\leq 1-\{\sum_{i=1}^{n} q^{d_i}\prod_{k=1}^{m_i}(1-q^{d_k-1})\prod_{k=m_i+1}^{i-1}(1-q^{d_k})\}$$

where $m_i=min(d_i, i\text{-}1)$, $i=1, 2, ... , n$.

For notational convenience, let

$$H(d_1,d_2,...,d_n)=$$

$$=1-\{\sum_{i=1}^{n} q^{d_i}\prod_{k=1}^{m_i}(1-q^{d_k-1})\prod_{k=m_i+1}^{i-1}(1-q^{d_k})\}.$$

$H(d_1, d_2, ... ,d_n)$ can be chosen as the upper bound of the reliability $R(G)$. Our goal is to find upper bound on $R(G_{m,\ l})$, $l=n=2$, $n+3$, $n(n+1)/2$. This problem can be transformed to find a network with degree sequence d_1, d_2, ... , and d_n such that $H(d_1, d_2, ... ,d_n)$ is maximal and

$$\sum_{i=1}^{n} d_i = 2l$$

holds.

Lemma 6 For any network $G(N, L, p)$ with degree sequence d_1, d_2, ... , and d_n, $(d_1\leq d_2\leq ...\leq d_n)$, if $d_s\angle d_{s+1}$ and $d_t\angle d_{t+1}$ where $s\angle t$, then there exist a node k with link $(k, t)\in L$ and link $(k, s)\notin L$. Remote link (k, t) from G and add link (k, s) to G. The resulting network G' has a greater upper bound. That is

$$H(d_1,...,d_s+1,...,d_t-1,...,d_n)\geq$$

$$\geq H(d_1,...,d_s,...,d_t,...,d_n).$$

Lemma 7 The network G with degree sequence $d_1,...,d_j$, $d_{j+1},...d_n=\alpha,...,\alpha, \alpha+1,..., \alpha+1$, where

$$\sum_{i=1}^{n} d_i = 2l \quad \text{maximizes}$$

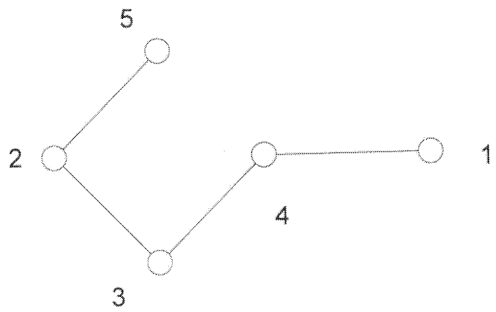
$H(d_1,...,d_j, d_{j+1},...d_n)$.

Over all degree sequences of the network with n nodes and l edges.

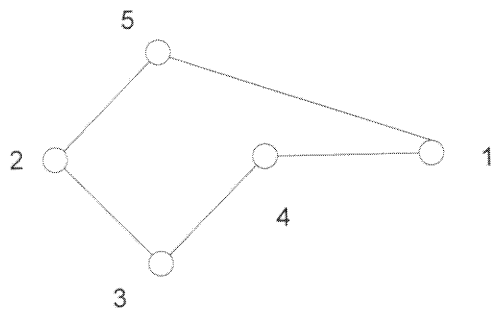
One interesting result from Lemma 7 is that the degrees of the nodes in any graph cannot differ by more than one in order for the graph to have a maximal value of $H(d_1, d_2, ... ,d_n)$. for example the network, the network with degree sequence 2, 2, 3, 3, 3, 3 has the maximal value of $H(d_1, d_2, ..., d_6)$ among all networks with degree sequence d_1, d_2, ..., d_n, where $d_1+d_2+...+d_6=16$.

4. COMPUTATIONAL RESULTS

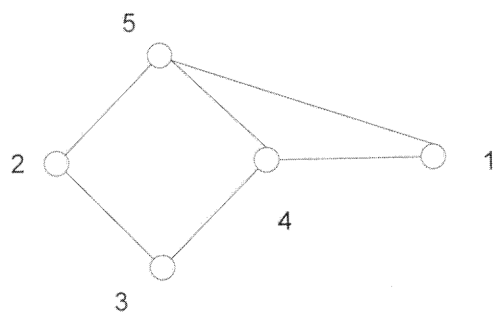
In the next graphs will be $p=0,9$, $n=5$



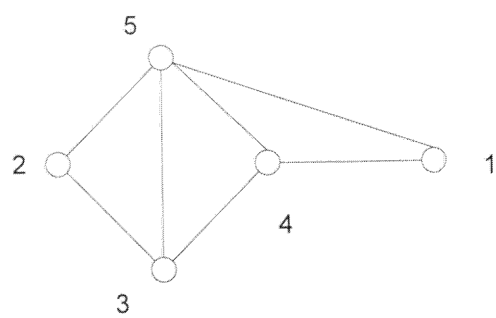
a) Tree network



b) To the tree network a) added the branch between nodes 1 and 5



c) To the network b) is added branch between nodes 1 and 4



d) To the network c) is added branch between nodes 3 and 5

Fig. 2 Example for calculations

In Fig. 2a is the tree with

$$R(G^{*}_{n,n-1})=p^4=0{,}9^4=0{,}6561.$$

In Fig. 2b is added the branch between nodes 1 and 5 and is

$$R(G^*_{n,n}) = p^n + np^{n-1}(1-p) = 0{,}91854.$$

In Fig. 2c is still added the branch between nodes 4 and 5 and is

$$R(G^*_{n,n+1}) = p^{n+1} + (n+1)p^n(1-p) +$$

$$+\frac{(n+1)^2}{3} p^{n-1}(1-p)^2 = 0{,}9536082.$$

In Fig. 2d is still added the branch between nodes 3 and 5 and is

node degrees $d_1 = d_2 = 2, d_3 = d_4 = 3, d_5 = 4$,

$$d_1 \le d_2 \le d_3 \le d_4 \le d_5$$

$$R(G) \le 1 - \{\sum_{i=1}^{n} q^{d_i} \prod_{k=1}^{m_i} (1-q^{d_k-1}) \prod_{k=m_i+1}^{i-1} (1-q^{d_k})\} =$$

$$= 0{,}9728900199,$$

where $m_i = min(d_i, i-1)$, $i=1, 2, \dots, n$.

The most effective rice of reliability is when to tree is added one branch.

5. CONCLUSIONS

This paper considers network topological optimization with a reliability constraint. The objective is to find the topological layout of links, at a minimal costs under the constraint that the network reliability is not less than a given level of system reliability. A decomposition method, based on branch and bound, is used for solving the problem. In order to speed-up the procedure, an upper bound on system reliability in terms of node degree is applied. A numerical example illustrates, and shows the effectiveness of the method.

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