GENERAL APPROACH TO COMPUTATION OF SELF- AND MUTUAL INDUCTANCES OF IRON-FREE COILS

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Summary The paper discusses various ways of computation of self- and mutual inductance of massive iron-free coils. These inductances are expressed by a multiple definite integral whose calculation is realized by the Monte Carlo (MC) method. The theoretical analysis is illustrated by four numerical examples.

1. INTRODUCTION

Various references describe a lot of ways of determining the self- and mutual inductances of iron-free coils. Classic works (let us mention, for example, books [1]–[3]) use empirical formulas and nomograms. Textbooks on electromagnetic fields introduce the self-inductance for thin loops by the *static* or *dynamic* definitions and for massive loops by the *energetic* definition. The mutual inductance is mostly given only for a pair of thin loops, and its value follows from the Neumann formula. For a pair of massive loops the mutual inductance is determined only exceptionally (see [4] and [5]). The published ways of calculation based on analytical (for instance [6]) or numerical ([7]–[10]) methods are applicable only for certain geometrical arrangements such as 2D. In papers [11] and [12] we showed that application of the Monte Carlo method allows determining inductances in much more general configurations. The method requires no special knowledge, does not have special demands on the computer capacity and the results can be obtained with an arbitrary accuracy.

The paper is aimed at elaboration of the Monte Carlo method, formulation of the computational algorithm even for mutual inductances of massive coils and investigation of how generation of random numbers affects velocity of computations.

2. GENERAL FORMULAS FOR COMPUTATION OF SELF- AND MUTUAL INDUCTANCES OF MASSIVE COILS

a) Self inductance. The self-inductance is given by relation

$$L = \frac{2W_{\mathrm{m}}}{I^2}, \tag{1}$$

where W_m is energy of magnetic field induced by current I. Energy W_m of magnetic field of a massive coil in linear medium (see Fig. 1) may be determined as,

$$W_{\mathrm{m}} = \frac{1}{2}\int_{V} \boldsymbol{H}\,\boldsymbol{B}\,\mathrm{d}V, \tag{2}$$

where V is the whole (unbounded) space. From the viewpoint of numerical computations it is more advantageous to express magnetic field energy by another formula.

$$W_{\mathrm{m}} = \frac{1}{2}\int_{V_{\mathrm{c}}} \boldsymbol{A}\,\boldsymbol{J}\,\mathrm{d}V, \tag{3}$$

where $\boldsymbol{A}$ is magnetic vector potential (that must be correctly normalized in order to avoid bad results) and $\boldsymbol{J}$ current density in the massive coil. Relation (3) is (with respect to (2)) much more advantageous because integration is now performed over a finite volume V_{c} of the coil.

Magnetic vector potential at an arbitrary point P inside the coil defined by position vector $\boldsymbol{r}_1$ (see, for example [4] and [5]) is expressed as

$$\boldsymbol{A}(P) = \frac{\mu_0}{4\pi}\int_{V_{\mathrm{c}}} \frac{\boldsymbol{J}(Q)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}\,\mathrm{d}V, \tag{4}$$

where $\boldsymbol{r}_2$ is the position vector defining an arbitrary point Q of volume V_c (Fig. 1).

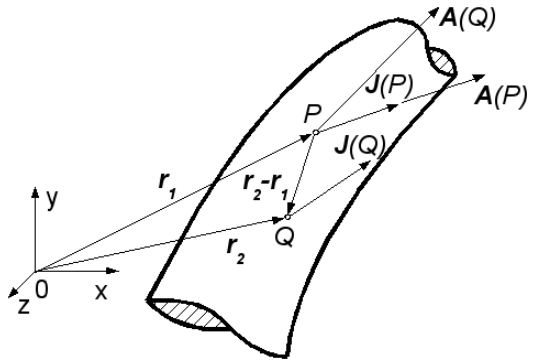

*Fig. 1. Vector potential **A** induced by current **J***

Equations (1), (3) and (4) provide the self-inductance in the form of a double volume integral over V_c:

$$L = \frac{\mu_0}{4\pi I^2}\int_{V_{\mathrm{c}}}\int_{V_{\mathrm{c}}} \frac{\boldsymbol{J}(P)\,\boldsymbol{J}(Q)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}\,\mathrm{d}V_{\mathrm{c}}\,\mathrm{d}V_{\mathrm{c}}. \tag{5}$$

Relation (5) holds for the self-inductance of a coil (or conductor) of an arbitrary geometrical

shape. Its numerical realisation carried out by some of the quadrature methods may become complicated due to singularity of the integrated function when $\boldsymbol{r}_1 = \boldsymbol{r}_2$. Then we have to cope with an improper integral. As we shall show later on, integration performed by means of the MC method avoids such a complication.

b) Mutual inductance. Consider a system of n massive coils carrying currents $I_1,\ldots,I_n$. Energy of magnetic field can be again expressed as

$$W = \frac{1}{2}\sum_{i=1}^{n}\int_{V_i} \boldsymbol{A}_i(\boldsymbol{r}_i)\boldsymbol{J}_i(\boldsymbol{r}_i)\,\mathrm{d}V\,, \tag{6}$$

where V_i is the region of the i-th coil, $\mathbf{r}_i \in \mathrm{V}_i$. Vector potential $\boldsymbol{A}_i(\boldsymbol{r}_i)$ is produced by currents in all n coils in the system, so that

$$\boldsymbol{A}_i(\boldsymbol{r}_i) = \frac{\mu_0}{4\pi}\sum_{j=1}^{n}\int_{V_j} \frac{\boldsymbol{J}_j(\boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\,\mathrm{d}V_j, \quad \boldsymbol{r}_j \in V_j \tag{7}$$

As vector potential $\boldsymbol{A}$ is not an unambiguous function, we norm it by putting $\boldsymbol{A}(\infty)=0$.

We can also remark that (7) implies magnetic flux density (*generalized Biot-Savart law*)

$$\boldsymbol{B}(\boldsymbol{r}) = \mathrm{rot}\,\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\sum_{i=1}^{n}\int_{V_i} \frac{\boldsymbol{J}(\boldsymbol{r}_i)}{|\boldsymbol{r} - \boldsymbol{r}_i|^3}\,\mathrm{d}V_i\,. \tag{8}$$

Equations (6) and (7) provide magnetic energy of a system containing n coils in the form

$$W_\mathrm{m} = \frac{\mu_0}{8\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{V_i}\int_{V_j} \frac{\boldsymbol{J}_i(\boldsymbol{r}_i)\boldsymbol{J}_j(\boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\,\mathrm{d}V_i\,\mathrm{d}V_j\,. \tag{9}$$

As known, magnetic energy of a system of n loops carrying currents $I_1,\ldots,I_n$ may be expressed as

$$W_\mathrm{m} = \sum_{i=1}^{n}\sum_{j=1}^{n} W_{\mathrm{m}ij} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}L_{ij}I_i I_j\,, \tag{10}$$

where $\mathrm{W}_{\mathrm{m}ij}$ and L_{ij} are partial energies and inductances (for $i=j$ self and for $i\neq j$ mutual).

Mutual inductances then follow from comparison of relations (9) and (10)

$$L_{ij} = \frac{\mu_0}{4\pi I_i I_j}\int_{V_i}\int_{V_j} \frac{\boldsymbol{J}_i(\boldsymbol{r}_i)\boldsymbol{J}_j(\boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\,\mathrm{d}V_i\mathrm{d}V_j\,, \tag{11}$$

Let us remark that for thin loops $\mathrm{c}_i,\mathrm{c}_j$ we get (using the limit transition) the *Neumann formula*

$$L_{ij} = \frac{\mu_0}{4\pi}\oint_{c_i}\oint_{c_j} \frac{\mathrm{d}\boldsymbol{l}_i\,\mathrm{d}\boldsymbol{l}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\,. \tag{12}$$

This formula can be used, however, only for $i=j$, because for $i\neq j$ the integral diverges.

3. FUNDAMENTALS OF THE MONTE CARLO METHOD

Let $f(\mathbf{x})$ be a real function defined on a k-dimensional bounded domain Ω, and we want to numerically calculate integral

$$I(f) = \int_{\Omega} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} f(x_1,\ldots x_k)\,\mathrm{d}x_1\ldots\mathrm{d}x_k \tag{13}$$

Suppose that the integral exists and reaches a finite value.

As the domain Ω is bounded, we can always find a rectangular domain K such that $\Omega \subset K$. The domain K is in particular dimensions bounded by intervals $[a_i,b_i], i=1,\ldots,k$. Then volume V_K is given by the product of lengths of particular intervals

$$V_K = \prod_{i=1}^{k}(b_i - a_i)\,. \tag{14}$$

Let $g(\mathbf{x})$ be another function defined on domain K such that

$$g(\boldsymbol{x}) = f(\boldsymbol{x}) \text{ for } \boldsymbol{x}\in\Omega \text{ and } 0 \text{ for } \boldsymbol{x}\in K-\Omega\,. \tag{15}$$

Now the values of the following integrals are equal

$$\int_K g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{16}$$

Denote letter $\boldsymbol{\xi}$ a k-dimensional random variable (random vector) such that $\boldsymbol{\xi}\in K$. The probabilistic function of density $h(\boldsymbol{x})$ in this region is a constant and vectors $\boldsymbol{\xi}$ in this region uniformly fill in the whole space $h(\boldsymbol{x})=c$. As the integral of any probabilistic function over the whole unbounded space K is equal to 1, there holds

$$c = V_K^{-1}\,. \tag{17}$$

Denote the value of function $g(\boldsymbol{\xi})$ for any random vector $\boldsymbol{\xi}$ by letter η, so that

$$g(\boldsymbol{\xi}) = \eta\,. \tag{18}$$

Then the assumed value of $E(\eta)$ of a random variable η is (by definition)

$$\begin{aligned} E(\eta) &= \int_K g(\boldsymbol{x})h(\boldsymbol{x})\,\mathrm{d}V = c\int_K g(\boldsymbol{x})\,\mathrm{d}K = \\ &= V_K^{-1}\int_{\Omega} f(\boldsymbol{x})\,\mathrm{d}\Omega = V_K^{-1}I(f). \end{aligned} \tag{19}$$

Hence

$$I(f) = V_K\,E(\eta)\,. \tag{20}$$

Relation (20) expresses the value of calculated integral as an assumed value of a certain random variable. This fact is of the fundamental importance for the MC method.

Consider n values $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\ldots,\boldsymbol{\xi}_n$ of random vector $\boldsymbol{\xi}$ and n corresponding function values

$\eta_i=g(\xi_i)$, $i=1,\ldots n$. Now the value of $E(\xi)$ may be approximated by the arithmetic average

$$\bar{\eta}(n) = \frac{1}{n}\sum_{i=1}^{n}\eta_i . \qquad (21)$$

It can easily be proved that $\bar{\eta}(n)$ converges to $E(\eta)$. For big values of n integral (13) can be approximated by relation

$$I(f) \approx V_K \bar{\eta}(n) . \qquad (22)$$

4. ERROR ESTIMATION OF THE MC METHOD

The derived relation says nothing about the convergence rate of $\bar{\eta}(n)$ to $E(\eta)$, i.e. to the correct value. Relation for intervals of reliability of integrals calculated by means of the MC method is derived in [11]. As the method is stochastic, its results may differ for different ways of calculation. The interval of reliability at the level of $k\,\%$ is an interval with $k\,\%$ of the obtained results. As we do not know in advance the exact value of the result, this interval can also be only estimated, but despite this fact it has an informative ability. For big values of n we may estimate the variance $\sigma^2(\eta)$ as

$$\sigma^2(\eta) \approx \frac{1}{n}\sum_{i=1}^{n}\eta_i^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\eta_i\right]^2 . \qquad (23)$$

On the basis of relation (23) we can estimate even $\bar{\eta}(n)$. Denote the standard deviation of $\bar{\eta}(n)$ and standard deviation of the whole integral by letter $\bar{\sigma}$:

$$\bar{\sigma}(n) = \sqrt{\frac{\sigma^2(n)}{n}} = \frac{1}{\sqrt{n}}\sigma(\eta) , \qquad (24)$$

where $\sigma(n)$ may be determined from (23).

If we use for finding of the interval of reliability $I(f)$ the normal distribution (for big n) of variable $\bar{\eta}(n)$, we will obtain the following intervals:

- for approximate probability 0.95 $\left(I(f)-2\bar{\sigma}, I(f)+2\bar{\sigma}\right)$
- and for approximate probability 0.997 $\left(I(f)-3\bar{\sigma}, I(f)+3\bar{\sigma}\right)$

5. METHOD QUASI-MONTE CARLO (QMC) – APPLICATION OF THE QUASI-RANDOM GENERATOR

The classic Monte Carlo method uses for generation of random numbers the *pseudo-random sequences*. An example is the linear congruent generator that can be described by means of a recurrent formula

$$x_n = \left(ax_{n-1}+c\right) \bmod m , \qquad (25)$$

where a,c,m are integers. The sequence is started from a selected value of x_0 (= seed). Outputs of these generators are random, but periodical, with a period less or equal to m. Any incorrect selection of the above parameters may, however, shorten the period. The disadvantage of these random sequences is that the groups of generated numbers exhibit certain geometrical structures, i.e. they do not uniformly fill in the whole domain, but they tend to form clusters and gaps.

From the principle of the MC method this fact is not advantageous. Using randomly generated vectors ξ we try to find the average value of function η, which can be at best achieved by uniform distribution of vectors ξ. One way is to use a regular grid. But its disadvantage may be that in case of irregular space it is often not easy to determine spacing of its points in order to obtain the required number of samples.

Instead of a uniform grid we can use, however, a generator that uniformly fills in the required space, called *quasirandom generator* (that belongs to the low discrepancy generators). These generators do not try to form random sequences and in this sense we cannot speak about the random generators. These sequences are deterministic that, without gaps and clusters, fill in the whole domain. The best known are the Corput, Halton and Fauer sequences.

The van der Corput series is given by reversion of the order of digits of the integer expressed at given basis b and back mapping into interval (0, 1). The integer at basis b may be expressed as

$$\sum_{k=0}^{q} a_k b^k , \qquad (26)$$

where q is the number of digits necessary for writing the number in the given system. Now we convert it into sum

$$\sum_{k=0}^{q} a_k b^{-k-1} . \qquad (27)$$

Example: Decimal number 13.

- In the binary system: 1101.
- Its reversion in the binary system: 1011.
- Its back decimal mapping into interval (0,1): 1011 = 1(1/2) + 0(1/4) + 1(1/8) + 1(1/16) = 11/16.

In this way number 1 generates 1/2, 10 generates 1/4, 11 generates 3/4 etc. The series of consecutive integers from 1 successively divides interval by halves and the resultant series is 1/2, 1/4, 3/4, 1/8, 3/8, 5/8,…. For an arbitrary number of points in one-dimensional case we obtain a grid with uniform step.

The Halton sequence represents an extension of the van der Corput sequence to higher dimensions. As we do not deal with the random numbers, we cannot repeatedly generate the numbers for obtaining multidimensional vectors and assign them to

particular dimensions of these vectors. In case of the Halton sequence we assign one Corput series to each dimension, but with different values of base b. For b we choose the prime numbers, usually as small as possible. So, for example, for generation of a three-dimensional vector of quasirandom numbers we use the van den Corput series with bases b_1=2, b_2=3, b_3=5 respectively. It can be shown that for smaller numbers of dimensions (<10) these series are usable, but in higher dimensions they exhibit clustering and irregular spacing. This lack can be removed by application of the Faur series. This series works on the same principle as the previous ones. For all dimensions we use the same prime number (mostly the nearest higher than the number of dimensions), but what is different is the back mapping into the decimal system.

6. NUMERICAL ALGORITHM OF COMPUTATION OF THE INTEGRAL USING MC OR QMC METHODS

The basic elements of the computation algorithm are the random (quasi- or pseudrandom) numbers in interval (0,1). Consider a domain Ω that is located in a rectangular domain K. The random vector $\boldsymbol{\xi}$ may be obtained either by repeated generation of particular components of the vector or by using, for instance, of the Halton sequence. The values from interval (0,1) will then be used for finding the components of vector $\boldsymbol{\xi}$ by means of formula

$$x_i = a_i + (b_i - a_i)\cdot p_i, \tag{28}$$

where x_i are the components of $\boldsymbol{\xi}$,, a_i and b_i are the lower and upper bounds of the interval determining the domain K and p_i is the randomly generated number from interval (0,1).

Computation of integral (5) or (11) starts from finding the average value of the integrated function in domain K, thus even beyond domain Ω where it is defined. Its average value is then calculated from the sum of values of $f(\boldsymbol{\xi})$. But if $\boldsymbol{\xi} \notin \Omega$, we put $f(\boldsymbol{\xi})$=0. For calculation of the interval of reliability we also determine the sum $f^2(\boldsymbol{\xi})$. The algorithm of computation may easily be described in the following manner:

```
sum=0
sum2=0
repeat N -times
        generate random vector ξ
        if ξ ∈ Ω, then
                sum = sum + f(ξ)
                sum2 = sum2+ f²(ξ)
        end if
end repeat
integral = V·sum/N
sigma= √((sum2/N − (sum/N)²/√N))
```

where V is the volume of rectangular domain K. In our case, when we determine the value of the double volume integral, the value V is the second power of the volume of the rectangular domain whose subdomain is the body of the coil. In this case vector $\boldsymbol{\xi}$ has six components, and both triples of them represent the elements of the real 3D domain.

7. EXAMPLES OF CALCULATION OF SELF- AND MUTUAL INDUCTANCES

The methods of MC and QMC integration were used for computation of two self- and two mutual inductances of coils depicted in Figs. 2, 3, 4 and 5.

Computations were always carried out for two generators of random numbers: pseudorandom numbers – a generator built in the translator of Fortran 90 for Linux F95 and quasirandom numbers – our own implementation of the Halton generator with bases 2, 3, 5, 7, 11 and 13. The computations were always performed for various numbers of repetitions. Simultaneously with the integral we calculated even the estimated interval of reliability at the level of 95%. The resultant graphs are depicted in Figs. 6 – 11. The algorithm was realized on a standard computer of type PC and the time of calculation even for the highest number of repetitions was of the order of seconds.

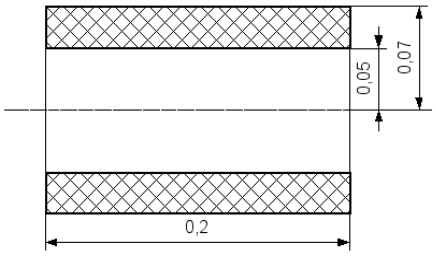

Fig. 2. Axisymmetric cylindrical coil from example 1

Example 1. Determination of selfinductance of a cylindrical coil in Fig. 2. The coil is massive, axisymmetric, the number of turns N=1.

The result is obvious from Fig. 6. This result was validated by another method using (1). The field energy was found by code FEMM for current I=10A and the magnetic field energy was $W_\mathrm{m} = 2.38\cdot10^{-6}$ J. The calculated self-inductance is then $L = \dfrac{2W_\mathrm{m}}{I^2} = 4.76\cdot10^{-8}$ H. This result is in a very good accordance with the value obtained for the highest number of repetitions (10^6) in the MC method: the pseudorandom generator provided inductance $4.78\cdot10^{-8}$ H and quasirandom generator $4.86\cdot10^{-8}$ H.

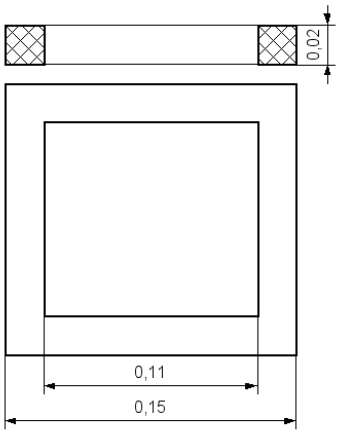

Fig 3. Coil from example 2

Example 2. Find the self-inductance of a rectangular coil according to Fig. 3. The coil is massive and contains one turn. The result is obvious from Fig. 7.

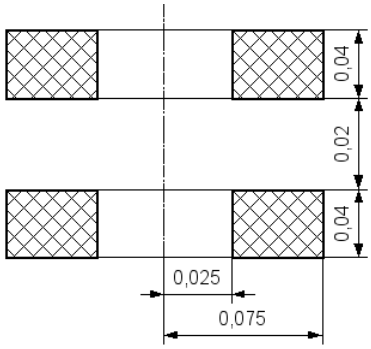

Fig. 4. Axisymmetric cylindrical coils from example 3

Example 3. Let us find mutual inductance of two coils. The coils are placed above one another and the whole arrangement is axisymmetric. The number of turns $N_1 = N_2 = 200$.

The results can be seen in Figs. 8 and 9. The result vas verified in the same manner as in the first example. The total magnetic energy of the system is $W_\mathrm{m} = W_\mathrm{m1} + W_\mathrm{m2} + W_\mathrm{m12}$. First we put field currents in both coils I_1=I_2=1A. Then we calculated energies $W_m(I_1,I_2)$=3.59mJ, W_{m1}=W_{m2}=1.4mJ and finally subtracted $W_m(I_1,I_2)$=0.79mJ. The mutual inductance is then M_{12}=0.79mH. Even in this case we can declare almost total agreement with the results obtained by the MC method for the number of repetitions 10^6. The pseudorandom generator provided 0.79mH and quasirandom generator 0.78mH.

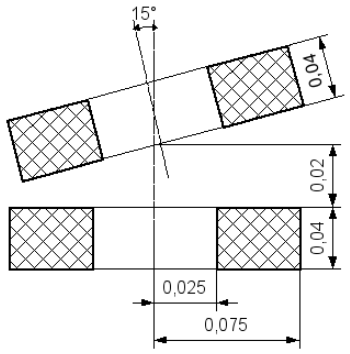

Fig. 5. Two axisymmetric cylindrical coils in general position from example 4

Example 4. Let us find mutual inductance of two coils that are in a general position above one another. One of them is turned through an angle of φ=15°.

The results are depicted in Figs. 10 and 11.

8. CONCLUSION

Various references (see, for example [13]) describe considerable advantages of the generators of quasirandom numbers in comparison with the generators of pseudorandom numbers. These advantages mainly consisted in higher rate of the convergence. But our examples did not confirm this knowledge. The graphs in Figs. 6–11 show that for the described applications the differences of the convergence rate are in both cases very small. For both generators it is obvious that the interval of reliability narrows down with the growing number of steps, which could be expected. The rate of narrowing is for both ways of generation practically the same and there is no reason for preferring any of them. But with respect to higher accessibility of the generator of pseudorandom numbers and practically the same convergence rate we can recommend it for computations of the indicated type.

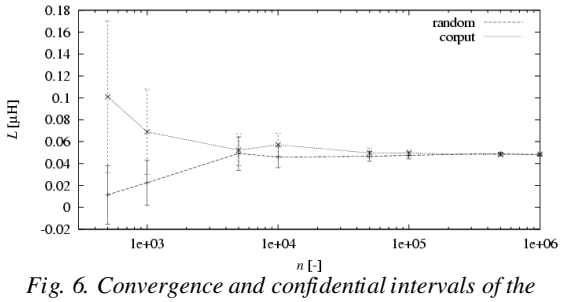

Fig. 6. Convergence and confidential intervals of the results from example 1

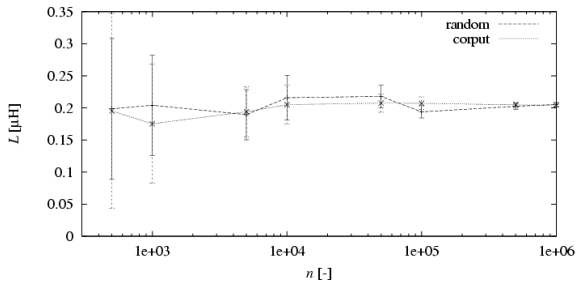

Fig. 7. Convergence and confidential intervals of the results from the example 2

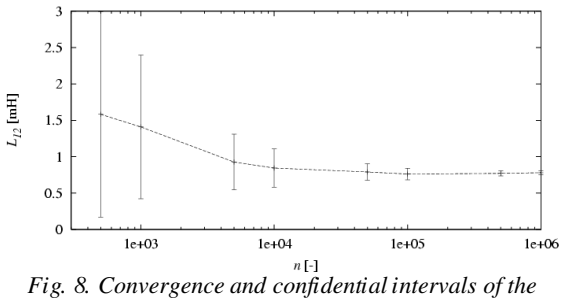

Fig. 8. Convergence and confidential intervals of the results from example 3 for the Corput sequence

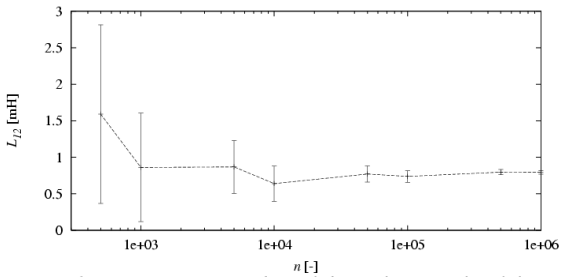

Fig. 9. Convergence and confidential intervals of the results from example 3 for the random sequence

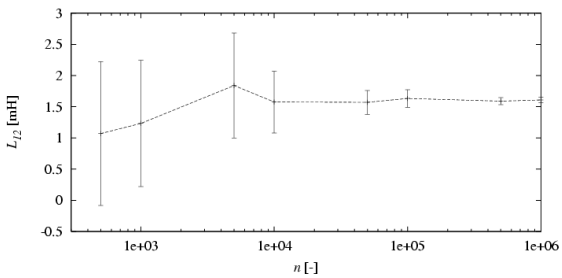

Fig. 10. Convergence and confidential intervals of the results from example 4 for the Corput sequence

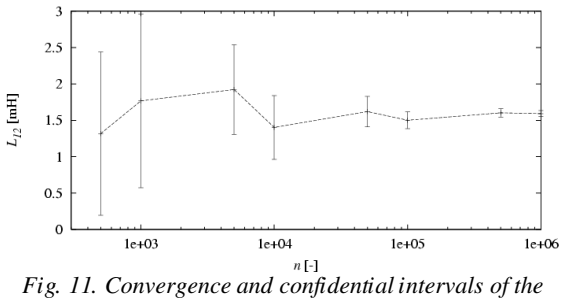

Fig. 11. Convergence and confidential intervals of the results from example 4 for the random sequence

Acknowledgement

This paper is based upon work sponsored by the Ministry of Education of the Czech Republic under Research Project MSM 4977751310 (Diagnostic of interactive action in electrical engineering).

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