TRANSITIONS OF MARKOVIAN PROCESS THROUGH A GIVEN LEVEL

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Summary This paper is a free continuation of [1] where properties of Markovian process were also dealt with. It may be necessary to determine the count of transitions through a given level and the mean time during that the random process persists over or below a chosen level. Traffic load estimations and predictions in dynamically controlled broadband networks may serve as an example of practical applications. Unlike of [1] where as many changes in the state of random process as possible shall be caught up, now we are only interested in those transitions which cross a chosen level.

Abstrakt Tento článok voľne nadväzuje na [1], v ktorom bolo pojednané aj o vlastnostiach Markovovho procesu. Niekedy je potrebné stanoviť počet prechodov procesu cez zadanú hladinu a strednú dobu, počas ktorej náhodný proces zotrváva nad alebo pod zvolenou hladinou. Ako príklad praktického využitia môžu poslúžiť odhady a predpovede prevádzkového zaťaženia v dynamicky riadených širokopásmových sieťach [2]. Na rozdiel od [1], kde sa musí podchytiť čo najviac zmien v stave náhodného procesu, sa teraz zaujímame iba o tie prechody, ktoré prejdú cez vybranú hladinu.

1. DEFINITIONS

Let denote:

λ - count of transitions from the free to the busy status of a non active traffic source during a time unit

μ - count of transitions from the busy to the free status of an active traffic source during a time unit

N – count of traffic sources offering traffic A

$a = \lambda/\mu$ - traffic load offered by 1 traffic source

$\bar{t} = 1/\mu$ - mean holding time

$x = t/\bar{t}$ - relative time

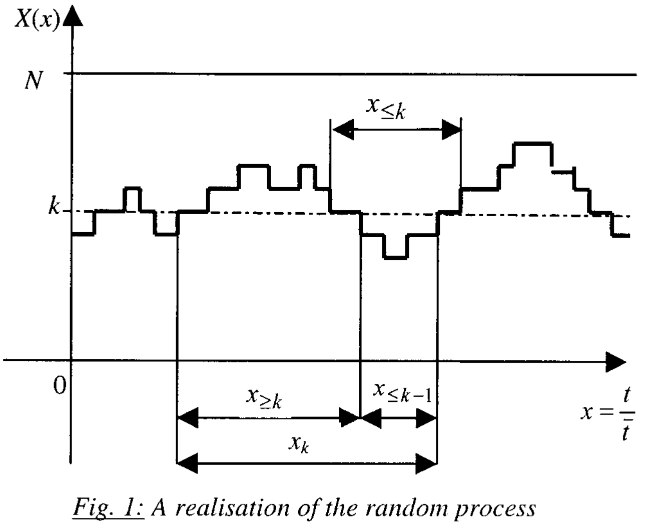

Fig. 1: A realisation of the random process

2. PERSISTENCE ON OR OVER A GIVEN LEVEL

Let 1 of possible realisations of the random process be considered (Fig. 1). Let $S_{\ge k}$ denote the status of the random process when k and more traffic sources are active. Let the transition to the status S_k when just k traffic sources are acti-ve happens in a moment. Then an arbitrary possible realisation of the process corresponding to the status when k and more traffic sources are active (k = 1,2,...,N), and being observed since that moment, only belongs to 1 type of possible trajectories ${}^1S_k$, ${}^2S_k$,...,${}^jS_k$ (Fig. 2). The trajectory ${}^1S_k$ corresponds to a random time interval ${}^1X_{k1}$ when just k traffic sources are active with the next transition under the given level k; the trajectory ${}^2S_k$ corresponds to a random time interval ${}^2X_{k1}$ when just k traffic sources are active with the consecutive transition over the given level k to the status when k and more traffic sources are active, and persisting in this status during a random time interval ${}^2Y_{k1}$ with the consecutive transition to the status when just k traffic sources are active during a random time interval ${}^2X_{k2}$ and with the next transition under the given level k. Generally, the trajectory ${}^jS_k$ corresponds to the realisation when the random process alternatively persists j – times in the status with just k active traffic sources and j-1 – times in the status with k and more active traffic sources (j = 1,2,...).

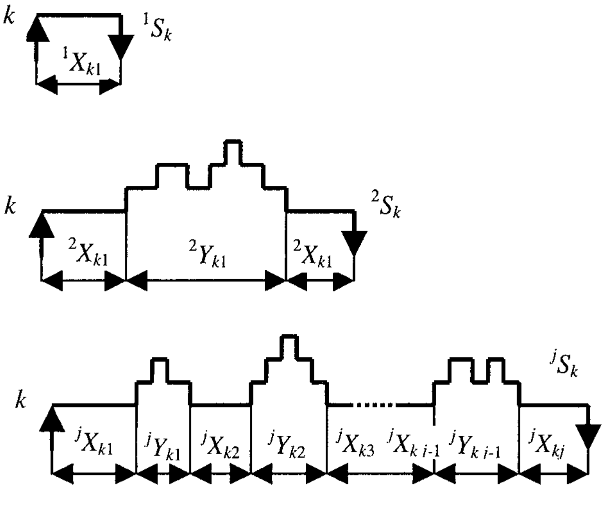

Fig. 2: Possible realisations of the random process

As the random process is Markovian, the random variables ${}^jX_{ki}$ (i = 1,2,...,j) and ${}^jY_{kl}$ (l = 1,2,...,j-1, j>0) are independent on each other whereby the random variables ${}^jX_{ki}$ have the same distribution function $P\{{}^jX_{ki}<t\}$ for all i = 1,2,...,j. This distribution function is given by (6) in [1]:

$$P\{{}^jX_{ki} < t\} = F_k(t) = 1 - e^{-a_{kk}t} \qquad i = 1,2,...,j, \; j>0 \tag{1}$$

Let the sought distribution function be denoted $P\{T_{\geq k}<t\}$, $k = 1,2,...N$. In order to derive it the total probability formula must be used:

$$P\{T_{\geq k}<t\} = \sum_{j=1}^{\infty} P\{^{j}S_k\}.P\{T_{\geq k}<t/^{j}S_k\} \tag{2}$$

where $P\{T_{\geq k}<t/^{j}S_k\} = F_{\geq k}(t/^{j}S_k)$ is the conditioned distribution function of the random time interval when k and more traffic sources are active and when the trajectory of the random process belongs to the realisation of the $^{j}S_k$ type, and $P\{^{j}S_k\}$ is the probability that this type of realisation will occur, so that:

$$\sum_{j=1}^{\infty} P\{^{j}S_k\} = 1 \qquad k = 1,2,...,N$$

If the status of k active traffic sources has occured by the transition from below, then the status $^{j}S_k$ will happen when the random process transits j-1 - times from the status with just k active traffic sources to the status with k and more active traffic sources and lives once the status with just k active traffic sources and then comes back below. The probability of that is:

$$P\{^{j}S_k\} = p_{k\,k+1}^{j-1} \cdot p_{k\,k-1}^{1}$$

Then (2) can be written in the form:

$$P\{T_{\geq k}<t\} = p_{k\,k-1}\sum_{j=1}^{\infty} p_{k\,k+1}^{j-1}.F_{\geq k}(t/^{j}S_k) \tag{3}$$

The probability $p_{k\,k+1}$ is the transition probability from the status with k active traffic sources to the higher status with more then k active traffic sources and the probability $p_{k\,k-1}$ is the transition probability from the status with k active traffic sources to the lower status with less than k active traffic sources. These are, in fact, the leaving probabilities from the status with k active traffic sources upwards and downwards in the stable state of the random process. They are given by (10) in [1]:

$$p_{k\,k+1} = \frac{a_{k\,k+1}}{a_{kk}} \tag{4}$$

$$p_{k\,k-1} = \frac{a_{k\,k-1}}{a_{kk}} \tag{5}$$

According to (19), (20), (21) in [1] it is:

$$a_{kk} = (N-k)\lambda + k\mu = \mu\left[(N-k)\frac{\lambda}{\mu} + k\right] = \frac{A+k}{\bar{t}}$$

$$a_{k\,k+1} = (N-k)\lambda = \mu(N-k)\frac{\lambda}{\mu} = \frac{A}{\bar{t}}$$

$$a_{k\,k-1} = k\mu = \frac{k}{\bar{t}}$$

where:

$A = a(N-k)$

Substituting in (1), (4), (5) we have:

$$P\{^{j}X_{ki}<x\} = F_k(x) = 1 - e^{-(A+k)x} \tag{6}$$

$$p_{k\,k+1} = \frac{A}{A+k} \tag{7}$$

$$p_{k\,k-1} = \frac{k}{A+k} \tag{8}$$

In order to find the mean relative time of persisting the process in the status with k and more active traffic sources $x_{\geq k}$ it is not necessary to completely know its corresponding distribution function $P\{T_{\geq k}<t\}$. Knowledge of generation function of this distribution will do. The generation function is defined as the mean of the random variable $W = e^{-sX}$ [3]:

$$G(s) = \overline{W} = \overline{e^{-sX}} = \int_0^{\infty} f(x)e^{-sx}dx \qquad \Re\{s\} > 0,\ t \geq 0$$

where $f(x)$ is the distribution of the random variable X.

Generation function of the sum of independent random variables is equal to the product of generation functions of these random variables:

$$G(s) = \prod_{i=1}^{j} G_i(s)$$

If all random variables have the same distributions, their generation functions are equal and then:

$G(s) = G^{j}_{i}(s)$

The conditioned distribution function $P\{T_{\geq k}<t/^{j}S_k\} = F_{\geq k}(t/^{j}S_k)$ is the distribution function of the sum of j random variables $^{j}X_{k1}$, $^{j}X_{k2}$,..., $^{j}X_{kj}$ and j-1 random variables $^{j}Y_{k1}$, $^{j}Y_{k2}$, ..., $^{j}Y_{k\,j-1}$ that are independent.

The random variables $^{j}X_{ki}$, $i = 1,2,...,j$ have the same distribution functions given by (6). The corresponding distribution is:

$f_k(x) = F_k'(x) = (A+k).e^{-(A+k)x} \qquad x \geq 0$

The generation function will be:

$$G_{k_i}(s) = (A+k)\int_0^{\infty} e^{-(A+k)x}.e^{-sx}dx = \frac{A+k}{A+k+s}$$

The generation function of the sum of independent random variables $^{j}X_{ki}$, $i = 1,2,...,j$ with the same distribution is:

$$G_k(s) = G_{k_i}^j(s) = \left(\frac{A+k}{A+k+s}\right)^j \tag{8}$$

The random variables ${}^jY_{kl}$, $l = 1,2,...,j$-1 have also the same distribution functions $P\{{}^jY_{kl}<t\} = F_{\geq k+1}(t)$ which are unknown and have the same generation functions $G_{\geq k+1}(s)$. The generation function of the sum of j-1 independent random variables ${}^jY_{kl}$, $l = 1,2,...j$-1 will then be $[G_{\geq k+1}(s)]^{j-1}$. So the the generation function $G_{\geq k/j}(s)$ of the conditioned distribution $P\{T_{\geq k}<t/{}^jS_k\} = F(t/{}^jS_k)$ will the product of the generation function (8) and $[G_{\geq k+1}(s)]^{j-1}$:

$$G_{\geq k/j}(s) = \left(\frac{A+k}{A+k+s}\right)^j G_{\geq k+1}^{j-1}(s)$$

and the generation function $G_{\geq k}(s)$ of the sought distribution function (3) will be:

$$G_{\geq k}(s) = p_{k\,k-1}\sum_{j=1}^{\infty} p_{k\,k+1}^{j-1}.G_{\geq k/j}(s) =$$

$$= \frac{k}{A+k}\sum_{j=1}^{\infty}\left(\frac{A}{A+k}\right)^{j-1}\left(\frac{A+k}{A+k+s}\right)^j G_{\geq k+1}^{j-1}(s) =$$

$$= \frac{k}{A+k}.\frac{A+k}{A+k+s}\sum_{j=1}^{\infty}\left[\frac{A}{A+k}.\frac{A+k}{A+k+s}.G_{\geq k+1}(s)\right]^{j-1} =$$

$$= \frac{k}{A+k+s}\sum_{j=1}^{\infty}\left[\frac{A}{A+k+s}G_{\geq k+1}(s)\right]^{j-1}$$

As $|\,G(s)\,| < 1$ and $\Re\{s\} > 0$, the terms of the sum are always lower than 1, e.g. the sum performs the infinite geometrical sequence. So:

$$G_{\geq k}(s) = \frac{k}{A+k+s}.\frac{1}{1-\dfrac{A}{A+k+s}G_{\geq k+1}(s)} =$$

$$= \frac{k}{A+k+s-A.G_{\geq k+1}(s)}$$

Derivation of generation function according s for $s = 0$ gives:

$$\left[\frac{dG(s)}{ds}\right]_{s=0} = \lim_{s\to 0}\frac{d}{ds}\int_0^{\infty} f(x)e^{-sx}dx = -\lim_{s\to 0}\int_0^{\infty} xf(x)e^{-sx}dx =$$

$$= -\int_0^{\infty} xf(x)dx = -m_1\{X\} = -\bar{x}$$

Then:

$$x_{\geq k} = -\left[\frac{dG_{\geq k}(s)}{ds}\right]_{s=0} = -\lim_{s\to 0}\frac{d}{ds}\left[\frac{k}{A+k+s-A.G_{\geq k+1}(s)}\right] =$$

$$= -\lim_{s\to 0}\frac{-k.[1-A.G'_{\geq k+1}(s)]}{[A+k+s-A.G_{\geq k+1}(s)]^2} = \frac{k.[1-A.G'_{\geq k+1}(0)]}{[A+k-A.G_{\geq k+1}(0)]^2} =$$

$$= \frac{k.(1+A.x_{\geq k+1})}{k^2} = \frac{1}{k} + \frac{A}{k}x_{\geq k+1} \tag{9}$$

Here:

$$G_{\geq k+1}(0) = \int_0^{\infty} f_{\geq k+1}(x).e^{-0.x}dx = \int_0^{\infty} f_{\geq k+1}(x)dx = 1$$

$$G'_{\geq k+1}(0) = -x_{\geq k+1}$$

The term (9) performs recurrent formula where $x_{\geq k}$ can implicitly be expressed as follows:

$$x_{\geq N+1} = 0$$

$$x_{\geq N} = x_N = \frac{1}{N} + \frac{A}{N}x_{\geq n+1} = \frac{1}{N}$$

$$x_{\geq N-1} = \frac{1}{N-1} + \frac{A}{N-1}x_N = \frac{1}{N-1} + \frac{A}{N(N-1)}$$

$$x_{\geq N-2} = \frac{1}{N-2} + \frac{A}{N-2}x_{\geq N-1} =$$

$$= \frac{1}{N-2} + \frac{A}{N-2}\left[\frac{1}{N-1} + \frac{A}{N(N-1)}\right] =$$

$$= \frac{A^0}{N-2} + \frac{A^1}{(N-1)(N-2)} + \frac{A^2}{(N-0)(N-1)(N-2)}$$

In general:

$$x_{\geq N-j} = \frac{A^0}{N-j+0} + \frac{A^1}{(N-j+1)(N-j)} +$$

$$+ \frac{A^2}{(N-j+2)(N-j+1)N-j+0)} + ... +$$

$$+ \frac{A^i}{(N-j+i)(N-j+i-1)...(N-j+1)(N-j+0)}$$

$$+ ... + \frac{A^j}{N(N-1)(N-2)...(N-j+2)(N-j+1)(N-j)}$$

Generally, when each term is multiplied by

$$\frac{(N-j-1)!}{(N-j-1)!}$$

we will have:

$$x_{\geq N-j} = \sum_{i=0}^{j}\frac{A^i(N-j-1)!}{(N-j+i)!}$$

Substituting $k = N-j$ the mean relative time $x_{\geq k}$ will be expressed explicitly:

$$x_{\geq k} = (k-1)!\sum_{i=0}^{N-k}\frac{A^i}{(k+i)!} = (k-1)!\sum_{i=0}^{N-k}\frac{[a(N-k)]^i}{(k+i)!}$$

$$k = 1,2,\ldots,N \tag{10}$$

3. PERSISTENCE ON OR BELOW A GIVEN LEVEL

Let the random process (Fig. 1) be again considered. Let the transition to the status S_k when just k traffic sources are active happens in a moment. Then the possible realisation of the process corresponding to the status when k and less traffic sources are active ($k = 0,1,2,\ldots,N-1$), and being observed since that moment only belongs to 1 type of possible trajectories ${}^1S_k$, ${}^2S_k,\ldots,{}^jS_k$ (Fig. 3). The trajectory ${}^1S_k$ corresponds to a random time interval ${}^1X_{k1}$ when just k traffic sources are active with the next transition over the given level k; the trajectory ${}^2S_k$ corresponds to a random time interval ${}^2X_{k1}$ when just k traffic sources are active with the consecutive transition under the given level k to the status when k and less traffic sources are active, and persisting in this status during a random time interval ${}^2Z_{k1}$ with the consecutive transition to the status when just k traffic sources are active during a random time interval ${}^2X_{k2}$ and with the next transition over the given level k. Generally, the trajectory ${}^jS_k$ corresponds to the realisation when the random process alternatively persists j – times in the status with just k active traffic sources and j-1 – times in the status with k and less active traffic sources ($j = 1,2,\ldots$).

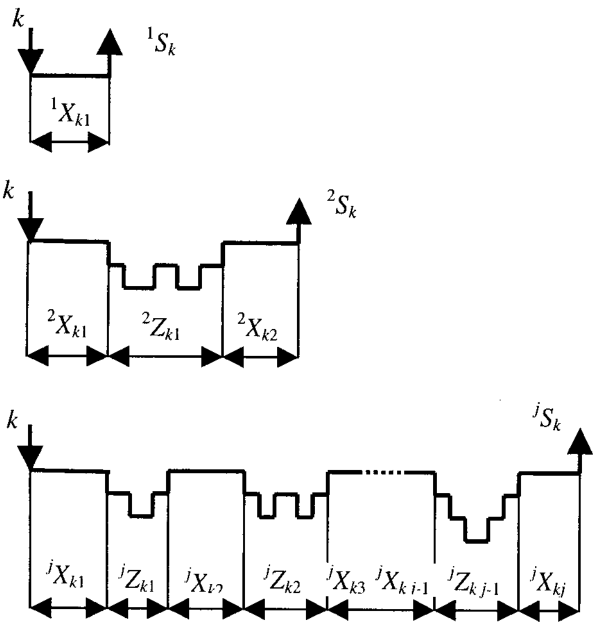

Fig. 3: Possible realisations of the random process

Let the sought distribution function be denoted $P\{T_{\le k}\}$, $k = 0,1,2,\ldots,N-1$. The procedure how to find the mean relative time $x_{\le k}$ is similar to the procedure like in Chapter 2:

$$P\{{}^jX_{ki} < t\} = F_k(t) = 1 - e^{-a_{kk}t} \qquad i = 1,2,\ldots,j,\ j>0$$

$$P\{{}^jZ_{kl} < t\} = F_{\le k-1}(t)$$

$$P\{T_{\le k} < t\} = \sum_{j=1}^{\infty} P\{{}^jS_k\}.P\{T_{\le k} < t/{}^jS_k\}$$

$$P\{T_{\le k} < t/{}^jS_k\} = F_{\le k}(t/{}^jS_k)$$

$$P\{{}^jS_k\} = p_{k\,k-1}^{j-1}.p_{k\,k+1}^{1}$$

$$P\{T_{\le k} < t\} = p_{k\,k+1}\sum_{j=1}^{\infty} p_{k\,k-1}^{j-1}.F_{\le k}(t/{}^jS_k)$$

$$G_{\le k/j}(s) = \left(\frac{A+k}{A+k+s}\right)^j G_{\le k-1}^{j-1}(s)$$

$$G_{\le k}(s) = p_{k\,k+1}\sum_{j=1}^{\infty} p_{k\,k-1}^{j-1}.G_{\le k/j}(s) =$$

$$= \frac{A}{A+k}\sum_{j=1}^{\infty}\left(\frac{k}{A+k}\right)^{j-1}\left(\frac{A+k}{A+k+s}\right)^j G_{\le k-1}^{j-1}(s) =$$

$$= \frac{A}{A+k}.\frac{A+k}{A+k+s}\sum_{j=1}^{\infty}\left[\frac{k}{A+k}.\frac{A+k}{A+k+s}.G_{\le k-1}(s)\right]^{j-1} =$$

$$= \frac{A}{A+k+s}\sum_{j=1}^{\infty}\left[\frac{k}{A+k+s}G_{\le k-1}(s)\right]^{j-1} =$$

$$= \frac{A}{A+k+s}.\frac{1}{1-\dfrac{k}{A+k+s}G_{\le k-1}(s)} =$$

$$= \frac{A}{A+k+s-k.G_{\le k-1}(s)}$$

$$x_{\le k} = -\left[\frac{dG_{\le k}(s)}{ds}\right]_{s=0} = -\lim_{s\to 0}\frac{d}{ds}\left[\frac{A}{A+k+s-k.G_{\le k-1}(s)}\right] =$$

$$= -\lim_{s\to 0}\frac{-A.[1-k.G'_{\le k-1}(s)]}{[A+k+s-k.G_{\le k-1}(s)]^2} = \frac{A.[1-k.G'_{\le k-1}(0)]}{[A+k-k.G_{\le k-1}(0)]^2} =$$

$$= \frac{A.(1+k.x_{\le k-1})}{A^2} = \frac{1}{A}+\frac{k}{A}x_{\le k-1}$$

Here:

$$G_{\le k-1}(0) = 1$$

$$G'_{\le k-1}(0) = -x_{\le k-1}$$

$$x_{\le -1} = 0$$

$$x_{\le 0} = \frac{1}{A}$$

$$x_{\le 1} = \frac{1}{A}+\frac{1}{A}.\frac{1}{A} = \frac{1}{A}\left(1+\frac{1}{A}\right)$$

$$x_{\leq 2} = \frac{1}{A} + \frac{2}{A}.\frac{1}{A}\left(1 + \frac{1}{A}\right) = \frac{1}{A}\left(1 + \frac{2}{A^1} + \frac{2.1}{A^2}\right)$$

$$x_{\leq k} = \frac{1}{A}\left[1 + \frac{k}{A} + \frac{k(k-1)}{A^2} + \frac{k(k-1)(k-2)}{A^3} + \ldots + \right.$$

$$+ \frac{k(k-1)(k-2)\ldots(k-i+1)}{A^i}.\frac{(k-i)!}{(k-i)!} + \ldots +$$

$$\left. + \frac{k(k-1)(k-2)(k-3)\ldots 3.2.1}{A^k}\right]$$

$$x_{\leq k} = k!\sum_{i=0}^{k} \frac{1}{A^{i+1}(k-i)!} = k!\sum_{i=0}^{k} \frac{1}{[a(N-k)]^{i+1}(k-i)!} \tag{11}$$

4. MEAN COUNT OF TRANSITIONS THROUG A GIVEN LEVEL

It is evident from Fig. 1 and (10) and (11) that the mean relative time interval between 2 consecutive transitions upwards to the status with k and more active traffic sources and thereby crossing the given level from below is:

$$x_k = \frac{t_k}{\bar{t}} = x_{\geq k} + x_{\leq k-1} =$$

$$= (k-1)!\sum_{i=0}^{N-k} \frac{A^i}{(k+i)!} + (k-1)!\sum_{i=0}^{k-1} \frac{1}{A^{i+1}(k-1-i)!} =$$

$$= (k-1)!\left[\frac{1}{k!} + \frac{A^1}{(k+1)!} + \frac{A^2}{(k+2)!} + \ldots + \frac{A^{N-k}}{N!} + \right.$$

$$\left. + \frac{1}{A^1(k-1)!} + \frac{1}{A^2(k-2)!} + \ldots + \frac{1}{A^{k-2}2!} + \frac{1}{A^{k-1}1!} + 1\right] =$$

$$= \frac{(k-1)!}{A^k}\left[\frac{A^k}{k!} + \frac{A^{k+1}}{(k+1)!} + \frac{A^{k+2}}{(K+2)!} + \ldots + \frac{A^2}{2!} + \frac{A^1}{1!} + 1\right] =$$

$$= \frac{(k-1)!}{A^k}\sum_{i=0}^{N} \frac{A^i}{i!} = \frac{(k-1)!}{[a(N-k)]^k}\sum_{i=0}^{N} \frac{[a(N-k)]^i}{i!} \tag{12}$$

If N is large enough, the sum is e^A. The number of traffic sources N is very large in telecommunication networks in comparison to telecommunication channels k. Then k can be neglected with regard to N and

$$A \approx a.N$$

$$x_k \approx \frac{(k-1)!}{A^k}.e^A$$

The mean count of transitions through a given level is:

$$n_k \approx 2.\frac{\Theta}{t_k} = 2.\frac{\Theta}{\bar{t}.x_k} = \frac{\Theta}{\bar{t}}.2k.\frac{A^k}{k!}e^{-A} = \frac{2\Theta k}{\bar{t}}.\Pi(A)$$

(13)

where Θ is observation time. As can be seen, the mean count of transitions is proportional to Poisson distribution.

Maximum of (13) can be determined by derivation:

$$\frac{dn_k(A)}{dA} = \frac{d}{dA}\left(\frac{\Theta}{\bar{t}}2k.\frac{A^k}{k!}e^{-A}\right) = \frac{2k\Theta A^{k-1}}{\bar{t}.k!}e^{-A}(k-A) = 0$$

from where:

$$A = k$$

Maximal transition frequency will be obtained when the level $k = A$ and an integer value of A will be set in (13):

$$n_{k_{max}} = \frac{\Theta}{\bar{t}}.\frac{2A^{A+1}}{A!}e^{-A}$$

This result says that fluctuations of the random process are most frequent by the level k that statistically equals to the mean count of active traffic sources.

If N is large enough but k can not be neglected with regard to N, maximal count of transitions will be:

$$n_{k_{max}} = 2.\frac{\Theta}{\bar{t}.x_k} = \frac{\Theta}{\bar{t}}2aN.\frac{[a(1-a)N]^{aN}}{(aN)!}e^{-a(1-a)N} \tag{14}$$

where $k = a.N$ is expressed by means of (24) in [1].

The equation (14) is shown on Fig. 4 for $\Theta = \bar{t}$.

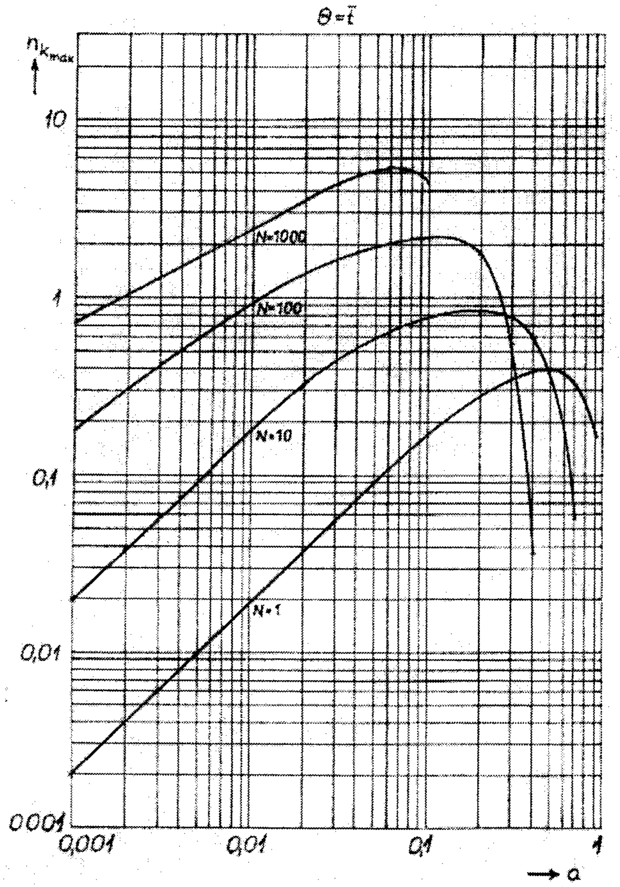

Fig.4: Maximal count of transitions through a given level

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