ELECTROMECHANICAL MODEL OF BLOOD FLOW IN VESSELS

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Summary. The present paper deals with some theoretical derivations connected with very efficient method of solution of hydrodynamic problems of blood flow in human cardiovascular system. The electromechanical analogy of liquid flow in a tube and electromagnetic wave propagating along an electric transmission line is discussed. We have derived a detailed circuit-like model of an elementary section of the elastic tube with viscose Newtonian liquid. The analogy harmonic current electrical circuit has been designed.

Keywords: electromechanical analogy, modelling and simulation, dynamic fluid systems, hemodynamics, equivalent electromagnetic system, transmission lines.

1. INTRODUCTION

In connection with development of computer technologies solutions of complex mathematical models of physical problems have taken place. One of such problems deals with computer aided simulation and investigation of physiological functions of a human organism [1]. The present paper focuses on a problem of modelling of processes, which take place in a human cardiovascular system. Theoretical analysis and computer aided modelling represent very useful and efficient tool for diagnostics of cardiovascular diseases and preparation of treatment [2], [3].

The paper deals with utilisation of electro-mechanical analogy for simulation of blood flow in blood vessels. The blood flow is formally similar to propagation of electric current along electric lines. Differential equations of hydrodynamic processes in a tube are similar to those describing transmission of electric charge in an electric line. Analogy of mechanical and electric quantities should be documented by Table 1.

Tab. 1

Mechanical quantity	**Electric quantity**
Liquid flow I^*	Electric current I
Mass density ρ	Inductivity L
Kinetic energy ½ $\rho\, v^2$	Magnetic energy ½ $L\, I^2$
Pressure p	Electric potential φ
Compliance k	Electric capacity C
Elastic energy ½ $k\, p^2$	Electric energy ½ $C\, U^2$
Viscosity η	Resistance R
Viscose losses $\eta\, v^2$	Electric losses $R\, I^2$
Wave resistance $\sqrt{\rho/k}$	Wave resistance $\sqrt{\mu/\varepsilon}$
Wave velocity $\sqrt{1/\rho k}$	Wave velocity $\sqrt{1/\mu\varepsilon}$

Analogy of hydrodynamic (mechanical) process in a tube and transmission of electric current can be demonstrated by schematic drawing in the Fig. 1.

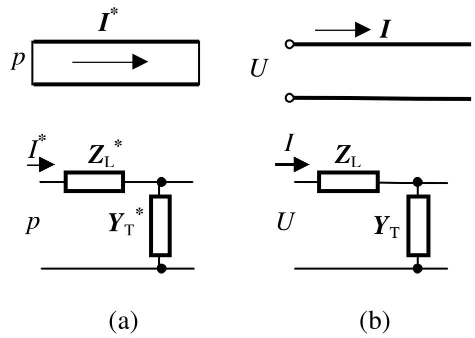

Fig. 1. Analogy of a tube with liquid and an electric line.
(a) Section of a homogeneous tube.
(b) Section of a homogeneous two-wire line.

Longitudinal impedances $\boldsymbol{Z}_L^*$ (mechanical) and $\boldsymbol{Z}_L$ (electric) represent conservative and dissipative components, transversal admittances $\boldsymbol{Y}_T^*$ and $\boldsymbol{Y}_T$ describe cross elasticity and inter-wire capacity and cross losses. All quantities are defined for a unit of length. In case of harmonic time dependences with angular frequency ω we obtain equations for complex amplitudes of the quantities

$$\frac{\partial \dot{p}}{\partial z} = -\boldsymbol{Z}_L^*\, \dot{I}^* \quad \leftrightarrow \quad \frac{\partial \dot{U}}{\partial z} = -\boldsymbol{Z}_L\, \dot{I} \qquad \text{(A)}$$

$$\frac{\partial \dot{I}^*}{\partial z} = -Y_T^*\, \dot{p} \quad \leftrightarrow \quad \frac{\partial \dot{I}}{\partial z} = -Y_T\, \dot{U}. \qquad \text{(B)}$$

These couples of equations lead to wave equations

$$\frac{\partial^2 \dot{I}^*}{\partial z^2} - \boldsymbol{k}^{*2}\, \dot{I}^* = 0 \qquad \frac{\partial^2 \dot{I}}{\partial z^2} - \boldsymbol{k}^2\, \dot{I} = 0, \qquad \text{(C)}$$

where complex wave numbers are

$$\boldsymbol{k}^* = \pm\sqrt{\boldsymbol{Z}_L^*\, \boldsymbol{Y}_T^*} \quad \text{or} \quad \boldsymbol{k} = \pm\sqrt{\boldsymbol{Z}_L\, \boldsymbol{Y}_T}\,. \qquad \text{(D)}$$

Taking real and imaginary parts into account

$\boldsymbol{k} = \pm(\alpha + \mathrm{j}\,\beta)$,

the direct wave of current or any other quantity depends on propagation coordinate z and time t as

$$\dot{P}(z,t) = \dot{P}_0\, e^{\mathrm{j}(\omega t - k z)} = \dot{P}_0\, e^{-\beta z}\, e^{\mathrm{j}(\omega t - \alpha z)}$$

and has a wavelength $\lambda = 2\pi/\alpha$ and coefficient of attenuation β.

We can see that solution of hydrodynamic processes in viscose liquids can be treated with formal tools of theory of electric circuits. Complex network of vessels can be taken as a network of parts of electric lines with different parameters. Such system can be solved by means of standard computer programs like MATLAB or others.

The main question of the analogy-based methodology consists in derivation of relationships for primary lines parameters $\boldsymbol{Z}_{\mathrm{L}}$ and $\boldsymbol{Y}_{\mathrm{T}}$.

2. PARAMETERS OF LIQUID FLOW

We shall investigate behaviour of liquid in an elastic tube. In order to simplify a general theory with respect of real conditions in most vessels we suppose non-turbulent flow of viscous liquid. Such liquid obeys the Navier-Stokes equation

$$\frac{\mathrm{d}(\rho \boldsymbol{v})}{\mathrm{d}t} = -\,\mathrm{grad}\; p + \eta\, \Delta \boldsymbol{v} + \boldsymbol{f} \quad , \qquad (1)$$

where ρ is the liquid density, $\boldsymbol{v}$ – local flow velocity, p – pressure, η - dynamic viscosity and $\boldsymbol{f}$ – external force density.

The additional condition of continuity of the liquid flow is

$$div(\rho\, \boldsymbol{v}) = -\frac{\partial \rho}{\partial t}. \qquad (2)$$

In our case we shall suppose an incompressible liquid with constant density ρ and all external forces will be neglected, $\boldsymbol{f} = 0$.

We shall describe a flow of liquid in a cylindrical elastic tube. Taking geometry of the system into account it is convenient to use cylindrical coordinates (r, φ, z).

Due to axial symmetry there is $v_\varphi = 0$ and the eq. (1) can be broken into only two components

$$\rho \frac{\partial v_r}{\partial t} = -\frac{\partial p}{\partial r} + \eta\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right)$$

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right).$$

Eq. (2) obtains a form

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} v_r + \frac{\partial v_z}{\partial z} = 0\,. \qquad (3)$$

Under practical conditions of long wavelength of pressure wave $\lambda >> r_0$ (r_0 is the tube radius) and because of it we can suppose

$$\frac{\partial^2 v_r}{\partial z^2} << \frac{\partial^2 v_r}{\partial r^2} \quad \text{and} \quad \frac{\partial^2 v_z}{\partial z^2} << \frac{\partial^2 v_z}{\partial r^2}\,.$$

After neglecting the smaller terms we obtain a system of two coupled linear equations

$$\begin{aligned} \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{1}{r^2} v_r - \frac{\rho}{\eta}\frac{\partial v_r}{\partial t} &= \frac{1}{\eta}\frac{\partial p}{\partial r} \\ \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} - \frac{\rho}{\eta}\frac{\partial v_z}{\partial t} &= \frac{1}{\eta}\frac{\partial p}{\partial z} \end{aligned} \qquad (4)$$

The system of partial differential equations will be solved for a case of harmonic time-dependence with angular frequency ω. We suppose a front wave in the tube so that the dependences of quantities should be expressed in the complex-functions form

$$\begin{aligned} \dot{p}(r,z,t) &= \dot{P}(r)\exp\left[\mathrm{j}(\omega t - k z)\right] \\ \dot{v}_r(r,z,t) &= \dot{V}_r(r)\exp\left[\mathrm{j}(\omega t - k z)\right] \\ \dot{v}_z(r,z,t) &= \dot{V}_z(r)\exp\left[\mathrm{j}(\omega t - k z)\right] \end{aligned} \qquad (5)$$

Substituting them in (3) and (4) we obtain ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}^2 \dot{V}_r}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\dot{V}_r}{\mathrm{d}r} - \left(\frac{1}{r^2} + \frac{\mathrm{j}\omega\rho}{\eta}\right)\dot{V}_r = \frac{1}{\eta}\dot{P}(r) \\ &\frac{\mathrm{d}^2 \dot{V}_z}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\dot{V}_z}{\mathrm{d}r} - \frac{\mathrm{j}\omega\rho}{\eta}\dot{V}_r = -\frac{\mathrm{j}k}{\eta}\dot{P}(r) \qquad (6) \\ &\frac{\mathrm{d}\dot{V}_r}{\mathrm{d}r} + \frac{1}{r}\dot{V}_r = \mathrm{j}k\,\dot{V}_z \end{aligned}$$

The first two are Bessel equations. Using a transformation $r \to a\, r$, where

$$a = \sqrt{-\frac{\mathrm{j}\omega\rho}{\eta}}\,, \qquad (7)$$

solutions of the Bessel equations have a form

$$\dot{V}_r(r) = A\, J_1\,(a\, r) + \dot{F}(r) \qquad (8)$$

$$\dot{V}_z(r) = B\, J_0\,(a\, r) + \dot{G}(r) \qquad (9)$$

where J_0 and J_1 are Bessel functions of the orders 0 and 1, A and B are proper constants, $\dot{F}(r)$ and $\dot{G}(r)$ are particular integrals respecting right sides of the differential equations.

The solutions must respect the third equation (6 c) – the binding condition. After substitution and utilization of basic properties of Bessel functions[1] we obtain

$$\frac{A}{B} = \frac{\mathrm{j}k}{a} \qquad (10)$$

and

$$\frac{\mathrm{d}\dot{F}}{\mathrm{d}r} + \frac{1}{r}\dot{F} = \mathrm{j}k\,\dot{G}. \qquad (11)$$

Substituting (9) into (6 b) we obtain a differential equation for the complex function $\dot{G}(r)$

$$\frac{\mathrm{d}^2\dot{G}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\dot{G}(r)}{\mathrm{d}r} - \frac{\mathrm{j}\omega\rho}{\eta}\dot{G}(r) = -\frac{\mathrm{j}k}{\eta}\dot{P}(r).$$

Variation of pressure across the tube is negligible

[1] $J_0'(x) = -J_1(x)$ and $x\, J_1'(x) = -J_1(x) + x\, J_0(x)$

and it means that $\dot{P}(r) \approx P$ is real constant. It means that the partial integral $\dot{G}(r)$ must be constant as well and we obtain

$$\dot{G}(r) = G = \frac{k}{\omega\rho} P . \tag{12}$$

Taking eq. (11) with the constant right side term into account, we obtain solution for the partial integral $\dot{P}(r)$, which has a form

$$\dot{P}(r) = \mathrm{j}\frac{k^2}{2\omega\rho}\ r\, P . \tag{13}$$

In addition we must take boundary conditions into account. Radial component of the fluid velocity at the tube wall must respect its movement caused by its elasticity. We suppose a wave of pressure (5) and corresponding modulation of the radius of the wall

$$\dot{r}_w(z,t) = r_0 + \Delta\dot{r}_0 \exp\left[\mathrm{j}(\omega t - k z)\right], \tag{14}$$

where r_0 is the undisturbed radius of the tube wall under an average pressure and $\Delta\dot{r}_0$ is complex amplitude of its modulation. The radial velocity at the tube wall is then

$$\dot{v}_r(r_0, z, t) = \frac{\mathrm{d}\dot{r}_w}{\mathrm{d}t} = \mathrm{j}\omega\Delta\dot{r}_0 \exp\left[\mathrm{j}(\omega t - k z)\right]$$

and

$$\dot{V}_r(r_0) = \mathrm{j}\omega\Delta\dot{r}_0 . \tag{15}$$

Because of symmetry the radial velocity is zero at the tube axis

$$\dot{V}_r(0) = 0 . \tag{16}$$

The longitudinal component of the fluid velocity reaches its maximum at the axis

$$\frac{\mathrm{d}\dot{V}_z(0)}{\mathrm{d}r} = 0 \tag{17}$$

and it is zero at the tube wall

$$\dot{V}_z(r_w) = 0 \tag{18}$$

because of adhesion of the fluid to the tube wall.

We shall now solve separately both components of the fluid velocity. At first we neglect the influence of the pressure wave on the boundary condition (18), which means $\dot{V}_z(r_w) \approx \dot{V}_z(r_0) = 0$. According to this condition we obtain, eq. (9),

$$B\, J_0(a r_0) + \frac{k}{\omega\rho} P = 0 .$$

The asked constants of the general solution are then

$$\begin{aligned} B &= -\frac{k}{\omega\rho}\,\frac{P}{J_0(a r_0)} \\ A &= -j\frac{k^2}{\omega\rho a}\,\frac{P}{J_0(a r_0)} \end{aligned} . \tag{19}$$

Resulting components of the fluid velocity are

$$\dot{V}_r(r) = \mathrm{j}\frac{k^2 r}{2\omega\rho}\left[1 - \frac{2 J_1(a r)}{a r\, J_0(a r_0)}\right] P \tag{20}$$

$$\dot{V}_z(r) = \frac{k}{\omega\rho}\left[1 - \frac{J_0(a r)}{J_0(a r_0)}\right] P . \tag{21}$$

The total fluid flow through the tube is[2]

$$\dot{Q} = \int_0^{r_0} \dot{V}_z(r)\, 2\pi r\, \mathrm{d}r = \mathrm{j}k\, \frac{\mathrm{j}\pi r_0^2}{\omega\rho}\,\frac{J_2(a r_0)}{J_0(a r_0)} P .$$

Taking (5) into account, we can write the fluid flow in a form

$$\dot{Q}(z,t) = -\,\mathrm{j}\frac{\pi r_0^2}{\omega\rho}\,\frac{J_2(a r_0)}{J_0(a r_0)}\,\frac{\partial \dot{p}(z,t)}{\partial z} . \tag{22}$$

If we compare this result with introductory relation (A), we obtain the longitudinal impedance

$$\boldsymbol{Z}_\mathrm{L}^{*} = -\frac{\mathrm{j}\omega\rho}{\pi r_0^2}\,\frac{J_0(a r_0)}{J_2(a r_0)} . \tag{23}$$

Next we take condition (15) into account

$$A\, J_1(a r_0) + \frac{\mathrm{j}k^2}{2\omega\rho} r_0 P = \mathrm{j}\omega\Delta\dot{r}_0 .$$

We suppose the linear relation between pressure and wall radius modulations

$$\dot{P} = \kappa \boldsymbol{K}\,\frac{\Delta\dot{r}_0}{r_0} , \tag{24}$$

where $\boldsymbol{K} = K_w + \mathrm{j}\,\omega\,\eta_w$, K_w is the volume stiffness of the tube wall and η_w – coefficient of internal friction (cause of deformation losses), κ - geometrical factor. Relation between pressure and change of radius can be expressed as

$$\dot{P} = -\,\mathrm{j}\omega\frac{2\pi r_0}{k^2}\left[-\frac{\mathrm{j}\omega\rho}{\pi r_0^2}\,\frac{J_0(a r_0)}{J_2(a r_0)}\right]\Delta\dot{r}_0 . \tag{25}$$

With respect of (D) and (24) we obtain transversal admittance

$$\boldsymbol{Y}_\mathrm{T}^{*} = \mathrm{j}\omega\frac{2\pi r_0^2}{\kappa\left(K_w + \mathrm{j}\omega\eta_w\right)} . \tag{26}$$

Using a linear model, we have derived components of the model parameters according to the Fig. 1.

3. CIRCUIT OF THE TUBE ELEMENT

Now we have to change terms (23) and (26) into electric-like circuits. At first we shall suggest design of the longitudinal element with the impedance $\boldsymbol{Z}_\mathrm{L}$ (eq. 23). We have a Bessel function of a complex variable. The most effective way to suggest a corresponding circuit structure consists in spreading out the term into a series

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[2] $\int_{x_1}^{x_2} x\, J_0(x)\, d x = \left[x\, J_1(x)\right]_{x_1}^{x_2}$ and

$$J_2(x) = -J_0(x) + \frac{2}{x} J_1(x) .$$

$$\boldsymbol{Z}_{\mathrm{L}}^{*} = -\frac{\mathrm{j}\omega\rho}{\pi r_0^2} \frac{\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{ar_0}{2}\right)^{2m}}{\left(\frac{ar_0}{2}\right)^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+2)!}\left(\frac{ar_0}{2}\right)^{2m}}$$

This expression can be written in the from of chain fraction

$$\boldsymbol{Z}_L^* = 2R_0 + \lambda R_0 + \cfrac{1}{\cfrac{3}{\lambda R_0} + \cfrac{1}{4R_0 + \cfrac{1}{\cfrac{5}{\lambda R_0} + \cfrac{1}{6R_0 + \Lambda}}}}$$

where $R_0 = \frac{4\eta}{\pi r_0^4}$ and $\lambda R_0 = \mathrm{j}\omega\frac{\rho}{\pi r_0^2} = \mathrm{j}\omega L_0$.

This fraction can be taken as the impedance of the structure designed in the Fig. 2.

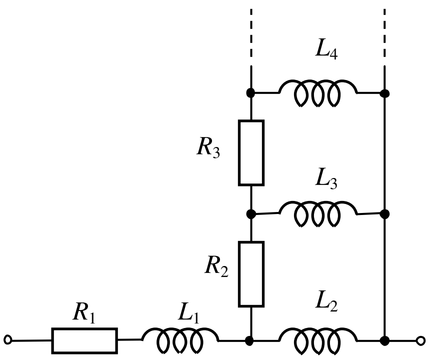

Fig. 2.

There is $R_n = 2n R_0$ and $L_n = \frac{1}{2n-1} L_0$.

Under real conditions the typical values of quantities are $\eta \approx 3.0$ mPa·s, $\rho \approx 1.1\times10^3$ kg·m^{-3}, $r_0 \approx 1.0$ mm, $f \approx 1.0$ Hz and then $R_0 \approx 3.8\times10^9$ Pa·s·m^4 and $\omega L_0 \approx 2.2\times10^9$ Pa·s·m^{-4}. Using only the direct line R_1, L_1 and L_2 we obtain the impedance $\mathbf{Z}_{L1}/R_0 \approx 2.144\ \angle 0.368$ rad. Taking the first bridge R_2, L_3 into account $\mathbf{Z}_{L2}/R_0 \approx 2.152\ \angle 0.367$ rad and with another bridge R_3, L_4 the difference occurs at the fifth digit. In means that the 5-elements circuit R_1, R_2, L_1, L_2 and L_3 is sufficiently accurate.

Parallel admittance (26), on the other side, is given by a simple expression, which can be modified as

$$\boldsymbol{Y}_{\mathrm{T}}^{*} = \frac{\mathrm{j}\omega C_{\mathrm{p}}}{1 + \mathrm{j}\omega C_{\mathrm{p}} R_{\mathrm{p}}},$$

where $C_{\mathrm{p}} = \frac{2\pi r_0^2}{\kappa K_w}$ and $R_{\mathrm{p}} = \frac{\kappa \eta_w}{2\pi r_0^2}$.

It represents a series of capacitor C_{p} and resistor R_{p}.

Resulting analogy circuit of the homogeneous section of a tube is in the Fig. 3.

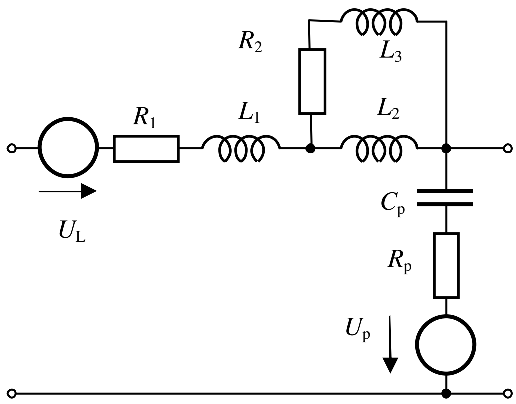

Fig. 3.

We took only alternating parts of hydrodynamic quantities and passive tube into account. There exist effects of active behaviour of blood vessels (supporting pumping mechanism) or additional constant pressure component of gravity and average blood pressure. These effects are added to the analogy circuit by means of serial source U_{L}. The parallel source U_{p} represents a constant tonus of blood vessels.

CONCLUSION

The derived analogy circuit is an image of a real blood segment and reflects all linear processes, which take place in it. The tree of blood system can be composed of such segments of different properties. There are computer methods of programming such systems and solving different dynamic processes. Another possibility of further development of the model consists in taking non-linear effects into account.

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